UNIVERSITÄT ZU LÜBECK
INSTITUTE FOR ROBOTICS
AND COGNITIVE SYSTEMS

Humanoid Robotics
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Nils Rottmann
rottmann@rob.uni-luebeck.de

Exercise sheet 1 - Kinematics

Please prepare the following exercises for the upcoming tutorial.

Task 1: Cart-Pole - Equation of Motion

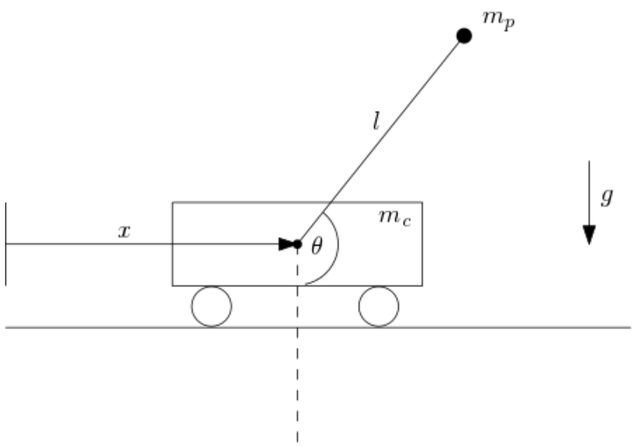

Figure 1 Cart-Pole

The in figure 1 depicted cart-pole shall be analyzed. Therefore the equation of motion have to be determined. We recommend to use the Lagrange equation. Further the so generated model shall be inserted into Matlab as a simulation model using the ODE solver. Afterwards, inverse kinematic shall be used in order to calculate the desired angle θ and position x of the cart-pole given a desired end effector position.

We start by determing the equation of motion. The kinematics of the system are given by

$$\boldsymbol{p}_c = \begin{bmatrix} x \\ 0 \end{bmatrix}, \quad \boldsymbol{p}_p = \begin{bmatrix} x + l\sin(\theta) \\ -l\cos(\theta) \end{bmatrix}. \tag{1}$$

There are different options to calculate the equation of motion. One would be to determine the force and torque balances for every body of the system. Feel free to test this on your own. You will see that every approach will lead to the same result. Here we will use the Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}, \quad L = T - V, \tag{2}$$

where T is the kinetic energy and V the potential energy. The generalized coordinates we are using are $q_1 = x$ and $q_2 = \theta$. Let us start by determining the kinetic energy T of the system:

$$T = \frac{1}{2}m_c\dot{x}^2 + \frac{1}{2}m_p(\dot{x}_p^2 + \dot{y}_p^2). \tag{3}$$

Here x_p and y_p are the coordinates of the pole. Transforming this equation using the above kinematics with

$$\dot{\boldsymbol{p}}_p = \begin{bmatrix} \dot{x}_p \\ \dot{y}_p \end{bmatrix} = \begin{bmatrix} \dot{x} + l\dot{\theta}\cos(\theta) \\ l\dot{\theta}\sin(\theta) \end{bmatrix} \tag{4}$$

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leads to

$$\begin{aligned} T &= \frac{1}{2}m_c\dot{x}^2 + \frac{1}{2}m_p\left[(\dot{x} + l\dot{\theta}\cos(\theta))^2 + (l\dot{\theta}\sin(\theta))^2\right] \\ T &= \frac{1}{2}(m_c + m_p)\dot{x}^2 + m_p l\dot{x}\dot{\theta}\cos(\theta) + \frac{1}{2}m_p l^2\dot{\theta}^2. \end{aligned} \tag{5}$$

The potential energy can be written as

$$V = -m_p gl\cos(\theta). \tag{6}$$

Together they lead to the Lagrange equation

$$L = \frac{1}{2}(m_c + m_p)\dot{x}^2 + m_p l\dot{x}\dot{\theta}\cos(\theta) + \frac{1}{2}m_p l^2\dot{\theta}^2 + m_p gl\cos(\theta). \tag{7}$$

We can now determine the derivatives according to the chosen generalized coordinates q_i which are required to determine the equation of motion:

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= (m_c + m_p)\dot{x} + m_p l\dot{\theta}\cos(\theta) \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} &= (m_c + m_p)\ddot{x} - m_p l\dot{\theta}^2\sin(\theta) + m_p l\ddot{\theta}\cos(\theta) \\ \frac{\partial L}{\partial \dot{\theta}} &= m_p l\dot{x}\cos(\theta) + m_p l^2\dot{\theta} \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} &= m_p l\ddot{x}\cos(\theta) - m_p l\dot{x}\dot{\theta}\sin(\theta) + m_p l^2\ddot{\theta} \\ \frac{\partial L}{\partial x} &= 0 \\ \frac{\partial L}{\partial \theta} &= -m_p l\dot{x}\dot{\theta}\sin(\theta) - m_p lg\sin(\theta). \end{aligned} \tag{8}$$

This leads to the equation of motion

$$\begin{aligned} (m_c + m_p)\ddot{x} - m_p l\dot{\theta}^2\sin(\theta) + m_p l\ddot{\theta}\cos(\theta) &= 0 \\ m_p l\ddot{x}\cos(\theta) - m_p l\dot{x}\dot{\theta}\sin(\theta) + m_p l^2\ddot{\theta} + m_p l\dot{x}\dot{\theta}\sin(\theta) + m_p lg\sin(\theta) &= 0. \end{aligned} \tag{9}$$

We can rewrite this in standard matrix notation as

$$\underbrace{\begin{bmatrix} m_c + m_p & m_p l\cos(\theta) \\ m_p l\cos(\theta) & m_p l^2 \end{bmatrix}}_{\boldsymbol{H}(\boldsymbol{q})} \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\ddot{\boldsymbol{q}}} + \underbrace{\begin{bmatrix} 0 & -m_p l\dot{\theta}\sin(\theta) \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})} \underbrace{\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}}_{\dot{\boldsymbol{q}}} + \underbrace{\begin{bmatrix} 0 \\ m_p lg\sin(\theta) \end{bmatrix}}_{\boldsymbol{G}(\boldsymbol{q})} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{10}$$

We can simulate this now in Matlab using ODE solver. With

$$x = 0, \quad \dot{x} = 0, \quad \theta = \frac{\pi}{2}, \quad \dot{\theta} = 0; \quad mc = 1, \quad mp = 1, \quad l = 1, \quad g = 9.81 \tag{11}$$

we will achieve the results depicted in 2.

Task 2: Cart-Pole - Inverse Kinematic

We want to steer the end effector, thus the pole of our system into a certain position. Therefore we need to use the inverse kinematic as presented in the lecture. First, we are using the geometric approach in order to calculate the desired angle θ and the position x. From equation 1 we got

$$\boldsymbol{p}_p = \begin{bmatrix} x + l\sin(\theta) \\ -l\cos(\theta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5l \end{bmatrix}. \tag{12}$$

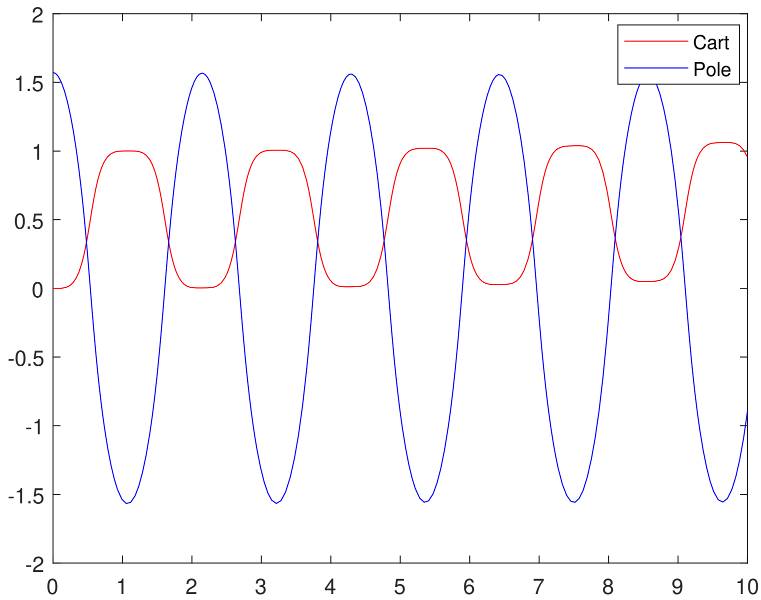

Figure 2 Cart Pole - Simulation Result

This yields

$$\theta = \cos^{-1}(-0.5) \tag{13}$$

and

$$x = 1 - l\sin(\theta) \tag{14}$$

which leads to

$$\theta = 2.0944, \qquad x = 0.1340. \tag{15}$$

Since it is not always useful to determine the desired positions for the joints analytically we now use numerical methods in order to determine the solution shown above. Therefore, assume the initial position of the end effector given as

$$\boldsymbol{p}_{p,0} = \begin{bmatrix} 0 \\ -l \end{bmatrix} \tag{16}$$

In order to determine the desired positions for the generalized coordinates, we will first use the Newton method. Thus, the Jacobian

$$J = \begin{bmatrix} \frac{\partial x_p}{\partial x} & \frac{\partial x_p}{\partial \theta} \\ \frac{\partial y_p}{\partial x} & \frac{\partial y_p}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 1 & l\cos(\theta) \\ 0 & l\sin(\theta) \end{bmatrix} \tag{17}$$

is required. With

$$\boldsymbol{p}_{p,d} = \begin{bmatrix} 1 \\ 0.5l \end{bmatrix} \tag{18}$$

as the desired position, we can define the error as

$$\boldsymbol{e} = \boldsymbol{p}_{p,d} - \boldsymbol{p}_p. \tag{19}$$

If the system would be linear, we could simply calculate

$$\Delta\boldsymbol{\theta} = J^{-1}\boldsymbol{e}, \tag{20}$$

but since our problem is non-linear, we have to iterate until the error $\boldsymbol{e}$ is sufficiently small. Therefore we can use Matlab. For the iteration we have to add a step size η which should be chosen small

$$\Delta\boldsymbol{\theta} = \eta J^{-1}\boldsymbol{e}. \tag{21}$$

Given a desired position for the end effector of $[1, 0.5l]^\top$, the iteration algorithm calculates $x = 0.1340$ and $\theta = 2.0944$.