

Exercise sheet 3 - Control Theory

Please prepare the following exercises for the upcoming tutorial.

Task 1: Closed-Loop PID Control

As an example throughout this exercise we will use a simple spring-damper model depicted in Figure 1 given the following parameters

$$m = 1, \quad b = 0.7, \quad k = 1. \quad (1)$$

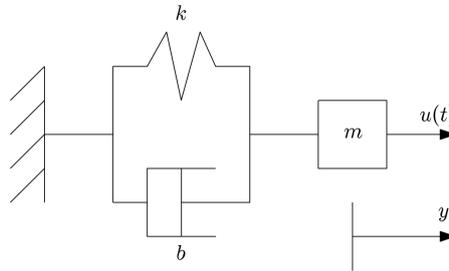


Figure 1 Spring-damper model

- a) The differential equation for the spring-damper system depicted in Figure 1 can be derived from the free body diagram given in Figure 2. Therefore we simply use Newton's equation of motion $F = ma$. This leads to

$$\ddot{y}(t)m = u(t) - F_k - F_b = u(t) - ky(t) - b\dot{y}(t). \quad (2)$$

Rewriting this equation yields

$$\ddot{y}(t) + \frac{b}{m}\dot{y}(t) + \frac{k}{m}y(t) = \frac{1}{m}u(t). \quad (3)$$

In the following we will use the following short hand notation $y = y(t)$, $u = u(t)$.

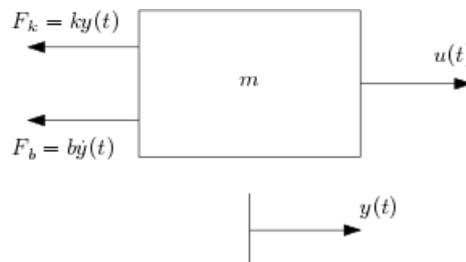


Figure 2 Free body diagram spring-damper system

- b) We can rewrite Equation (3) in a system of order 1 using $v = \dot{y}$. This leads to

$$\begin{bmatrix} \dot{y} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u. \quad (4)$$

c) The definition of a standard linear State-Space Model is given as

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \dot{\mathbf{y}} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u},\end{aligned}\quad (5)$$

where \mathbf{A} is called the system matrix. The first equation represents the dynamics of the plant with $\mathbf{x} \in \mathbb{R}^{n \times 1}$ as the state variables and is called the state equation. The second equation defines the plant output as a linear combination of state variables and input. It is called the output equation. The input into the system is defined as $\mathbf{u} \in \mathbb{R}^{m \times 1}$ and the output as $\mathbf{y} \in \mathbb{R}^{l \times 1}$. Rewriting equation 4 into such a State-Space Model yields

$$\begin{aligned}\begin{bmatrix} \dot{y} \\ \dot{v} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \\ y &= [1 \quad 0] \begin{bmatrix} y \\ v \end{bmatrix}.\end{aligned}\quad (6)$$

d) The Laplace Transform is defined as follows

$$\mathcal{L}[x(t)] = \int_0^{\infty} x(t)e^{-st} dt. \quad (7)$$

Here s is complex variable which is interpretable as a complex frequency. Moreover, the Laplace Transform is used to convert a differential equation from time domain into frequency domain. The so transformed model is represented by a transfer function and only algebraic operations are required. For example a convolution in time domain turns into a multiplication in frequency domain. For transform rules have a look into the appendix. As an example for the Laplace Transform we will use the following generalized differential equation

$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \dots + a_1\dot{y}(t) + a_0y(t) = b_m\frac{d^m}{dt^m}u(t) + \dots + b_1\dot{u}(t) + b_0u(t). \quad (8)$$

Using the Laplace Transform leads to

$$s^n Y(s) + a_{n-1}s^{n-1}Y(s) + \dots + a_1sY(s) + a_0Y(s) = b_ms^m U(s) + \dots + b_1sU(s) + b_0U(s). \quad (9)$$

Notice that differential equation which represents a real physical system has $n \geq m$. We can rewrite Equation 9 into

$$Y(s) = \frac{b_ms^m + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}U(s) \quad (10)$$

with

$$G(s) = \frac{b_ms^m + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0} \quad (11)$$

as the transfer function of the system. We can now transform the differential equation from (a). This leads to

$$Y(s) = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}U(s). \quad (12)$$

e) The State-Space Model from Equation 5 can also be transformed into frequency domain using the Laplace Equation which leads to

$$\begin{aligned}s\mathbf{X}(s) &= \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \rightarrow \mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \\ \rightarrow \mathbf{Y}(s) &= (\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D})\mathbf{U}(s).\end{aligned}\quad (13)$$

with

$$G(s) = C(sI - A)^{-1}B + D \quad (14)$$

as the transfer function of the system. The Laplace Transform of the in (c) generated system is given as

$$Y(s) = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}U(s). \quad (15)$$

which is the same as calculated above in (d).

- f) With the final value theorem the value for $x(t)$ as $t \rightarrow \infty$ can be calculated, if $x(t)$ converges to a finite and constant value. It states

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s). \quad (16)$$

The counterpart of the final value theorem is the initial value theorem. It can be used to find the value of $x(t)$ as $t \rightarrow 0^+$. Thus the limit when approaching zero from the right. It states

$$x(0^+) = \lim_{s \rightarrow \infty} sX(s). \quad (17)$$

Using the final value theorem to calculate the answer for a unit step response for the system given above yields

$$\begin{aligned} Y(s) &= \frac{1}{s^2 + 0.7s + 1}U(s) = \frac{1}{s^2 + 0.7s + 1} \frac{1}{s} \\ \rightarrow \lim_{t \rightarrow \infty} y(t) &= \lim_{s \rightarrow 0} s \frac{1}{s^2 + 0.7s + 1} \frac{1}{s} = \lim_{s \rightarrow 0} \frac{1}{s^2 + 0.7s + 1} = 1. \end{aligned} \quad (18)$$

- g) Consider a system with a transfer function given as

$$G(s) = \frac{b_m s^m + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}. \quad (19)$$

The values of s at which the denominator of $G(s)$ becomes zero, thus the values at which $G(s)$ becomes infinite, are called the poles of the transfer function $G(s)$. The poles for the transfer function calculated in (d) and (e) are given as

$$s_{1,2} = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - \frac{k}{m}}. \quad (20)$$

The location of the poles in the complex plane determine the dynamic behavior of the system, e.g. a system is stable, if all poles of the system are situated in the left half plane.

- h) One of the most important configurations for control theory is shown in Figure 3. Lets determine the closed-loop transfer function. Therefore we start with

$$Y_1(s) = G_1(s)R(s) - G_1(s)G_2(s)Y_1(s). \quad (21)$$

This leads to

$$(1 + G_1(s)G_2(s))Y_1(s) = G_1(s)R(s) \quad (22)$$

and with $G_{cl}(s)$ as the closed-loop transfer function to

$$G_{cl}(s) = \frac{G_1(s)}{1 + G_1(s)G_2(s)}. \quad (23)$$

In general we can write

$$\text{closed-loop transfer function} = \frac{\text{forward gain}}{1 + \text{loop gain}}. \quad (24)$$

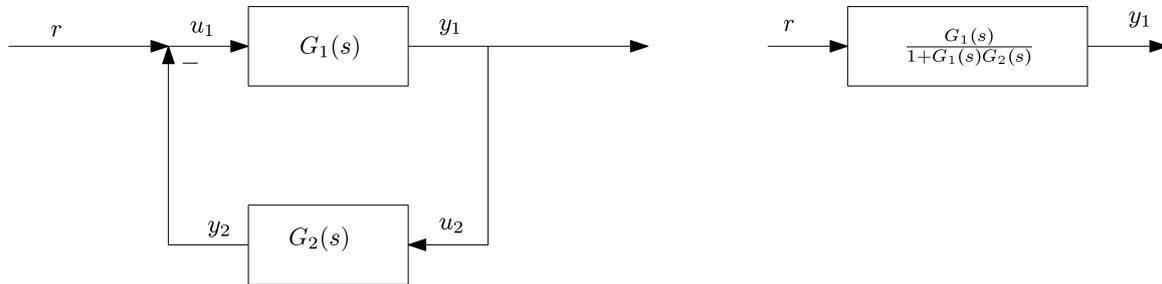


Figure 3 Closed Loop Configuration

- i) In Figure 4 a standard feedback control loop is shown. The tracking function can be derived as follows

$$Y = \frac{GC}{1 + GC}R \quad (25)$$

and the disturbance rejection transfer function as

$$Y = \frac{G}{1 + GC}D. \quad (26)$$

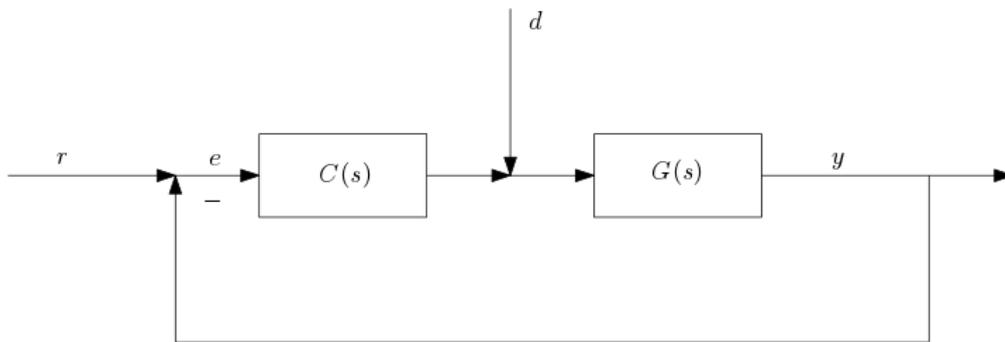


Figure 4 Closed-Loop Control Loop

- l) A general framework for a PID controller can be defined in time domain as

$$c(t) = K_P \left(e(t) + \frac{1}{T_I} \int_{t_0}^t e(\tau) d\tau + T_D \dot{e}(t) \right). \quad (27)$$

Transformation into frequency domain yields

$$C(s) = K_P \left(1 + T_D s + \frac{1}{T_I s} \right). \quad (28)$$

Figure 5 shows an example for the response of the spring-damper system introduced above disturbed by a unit step.

Task 2: LQR

- a) Given the equation determined in exercise 1

$$\begin{aligned} (m_c + m_p)\ddot{x} - m_p l \dot{\theta}^2 \sin(\theta) + m_p l \ddot{\theta} \cos(\theta) &= 0 \\ m_p l \ddot{x} \cos(\theta) + m_p l^2 \ddot{\theta} + m_p l g \sin(\theta) &= 0. \end{aligned} \quad (29)$$

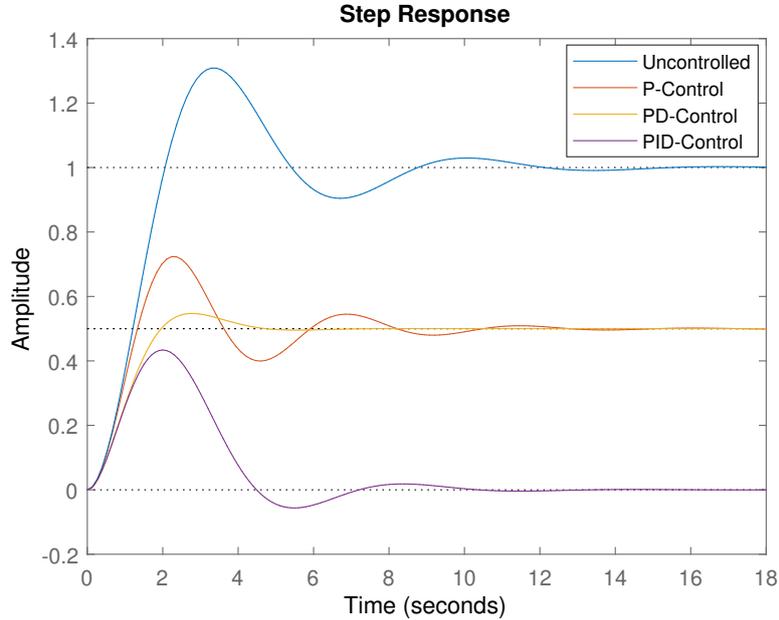


Figure 5 Example PID-Control

a linearization around $\theta = \pi + \phi$, together with the additional input term $u(t)$, yields

$$\begin{aligned} (m_c + m_p)\ddot{x} + m_p l \ddot{\phi} \cos(\phi) &= u \\ -m_p l \ddot{x} + m_p l^2 \ddot{\phi} + m_p l g \phi &= 0. \end{aligned} \quad (30)$$

There we used the following linearization procedure

$$\begin{aligned} \cos(\theta) &= \cos(\pi + \phi) \approx -1 \\ \sin(\theta) &= \sin(\pi + \phi) \approx -\phi \\ \dot{\theta}^2 &= \dot{\phi}^2 \approx 0. \end{aligned} \quad (31)$$

Rewriting the equation above into matrix form leads to

$$\underbrace{\begin{bmatrix} m_c + m_p & -m_p l \\ -m_p l & m_p l^2 \end{bmatrix}}_M \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\phi} \end{bmatrix}}_{\ddot{\mathbf{y}}} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -m_p g l \end{bmatrix}}_K \underbrace{\begin{bmatrix} x \\ \phi \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\mathbf{f}} u. \quad (32)$$

The State-Space-System is given as

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{y}} \\ \ddot{\mathbf{y}} \end{bmatrix} &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -M^{-1}K & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \dot{\mathbf{y}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ M^{-1}\mathbf{f} \end{bmatrix} u. \\ x &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \dot{\mathbf{y}} \end{bmatrix}. \end{aligned} \quad (33)$$

- b) The system is stable if and only if all poles of the transfer function are in the left-half of the complex plane. The above cart-pole around the point $\theta = \pi$ is unstable, which is also implied by calling this point

the unstable equilibrium from the cart-pole.

A system is controllable if and only if the controllability matrix $\mathcal{C}(\mathbf{A}, \mathbf{B})$ has the full rank. The controllability matrix is given as

$$\mathcal{C}(\mathbf{A}, \mathbf{B}) = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}] \quad (34)$$

where n is the number of states of the system. If a system is controllable than it is possible to find a control input that takes the system from any initial state to any final state in any given time interval. We can check with the help of Matlab that our given system is controllable.

- c) The idea of a LQR controller is to find a feedback control law

$$\mathbf{u} = -\mathbf{K}\mathbf{x} \quad (35)$$

which minimizes the cost given by

$$J = \int_{t_0}^{t_1} (\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u}) dt. \quad (36)$$

A state-feedback with such an controller is shown in 6.

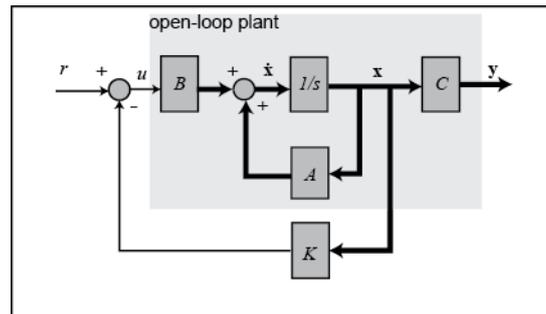


Figure 6 State-Feedback with LQR

- d) For a detailed solution have a look into the Matlab File for this Exercise.